AN EXPLICIT FORMULA FOR A STAR PRODUCT WITH SEPARATION OF VARIABLES

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Dedicated to the memory of Nikolai Neumaier

Abstract. For a star product with separation of variables $*$ on a pseudo-Kähler manifold we give a simple closed formula of the total symbol of the left star multiplication operator L_f by a given function f. The formula for the star product $f * g$ can be immediately recovered from the total symbol of L_f.

1. Introduction

Given a vector space W and a formal parameter ν, we denote by $W[[\nu]]$ the space of formal vectors $w = w_0 + \nu w_1 + \nu^2 w_2 + \ldots, w_r \in W$. One can also consider formal vectors that are formal Laurent series in ν with a finite polar part,

$$w = \sum_{r \geq k} \nu^r w_r$$

with $k \in \mathbb{Z}$.

Let M be a Poisson manifold endowed with a Poisson bracket $\{\cdot, \cdot\}$. A star product $*$ on M is an associative product on the space $C^\infty(M)[[\nu]]$ of formal functions on M given by a ν-adically convergent series

$$f * g = \sum_{r=0}^{\infty} \nu^r C_r(f, g),$$

where C_r are bidifferential operators, $C_0(f, g) = fg$, and $C_1(f, g) - C_1(g, f) = i\{f, g\}$ (see [1]). We also assume that the unit constant is the unity of the star-product $*$. A star product can be restricted to an open subset of M and recovered from its restrictions to subsets forming an open covering of M. Given functions $f, g \in C^\infty(M)[[\nu]]$, denote by L_f and R_g the left star multiplication operator by f and the right star multiplication by g, respectively. Then $L_f g = f * g = R_g f$ and the

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associativity of $*$ is equivalent to the property that $[L_f, R_g] = 0$ for any f, g. The operators L_f and R_g are formal differential operators on M. It was proved by Kontsevich in [9] that deformation quantizations exist on arbitrary Poisson manifolds.

A star product is called natural if, for each r, the bidifferential operator C_r is of order not greater than r in each of its arguments (see [6]). We call a formal differential operator $A = A_0 + \nu A_1 + \nu^2 A_2 + \ldots$ natural if the order of A_r is not greater than r. If a star product is natural, the operators L_f and R_f for any $f \in C^\infty(M)[[\nu]]$ are natural. The star products of Fedosov [4] and Kontsevich [9] are natural.

Now let M be a pseudo-Kähler manifold of complex dimension m endowed with a pseudo-Kähler form ω_{-1} and the corresponding Poisson bracket $\{\cdot, \cdot\}$. A star product with separation of variables $*$ on M is a star product such that the bidifferential operators C_r differentiate the first argument in antiholomorphic directions and the second argument in holomorphic ones (see [7], [3]). Star products with separation of variables appear naturally in the context of Berezin quantization (see [2]). It was proved in [3] and [8] that the star products with separation of variables are natural in the sense of [6].

A star product on a pseudo-Kähler manifold M is a star product with separation of variables if and only if for any local holomorphic function a and a local antiholomorphic function b on M the operators L_a and R_b are pointwise multiplication operators by the functions a and b, respectively,

$$L_a = a, \ R_b = b.$$

Otherwise speaking, if f is a local holomorphic or g is a local antiholomorphic function, then $f * g = fg$.

A formal form $\omega = \frac{1}{\nu}\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ such that the forms $\omega_r, r \geq 1$, are of type (1,1) with respect to the complex structure on M and may be degenerate is called a formal deformation of the pseudo-Kähler form ω_{-1}. It was proved in [7] that the star products with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}) are bijectively parametrized by the formal deformations of the form ω_{-1} (see also [10]).

A star product with separation of variables $*$ on (M, ω_{-1}) corresponds to a formal deformation ω of the form ω_{-1} if for any contractible holomorphic chart $(U, \{z^k, \bar{z}^l\})$, where $1 \leq k, l \leq m$, and a formal potential $\Phi = \frac{1}{\nu}\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots$ of ω (i.e., $\omega = i\partial\bar{\partial}\Phi$) one has

$$R_{\nu \frac{\partial \Phi}{\partial \bar{z}^l}} = \nu \left(\frac{\partial \Phi}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l} \right).$$

The star product with separation of variables $*$ parametrized by a given deformation ω of ω_{-1} can be constructed as follows. As shown in [7],

for any formal function f on U one can find a unique formal differential operator A on U commuting with the operators $R_{\bar{z}^l} = \bar{z}^l$ and $R_{\nu \frac{\partial \Phi}{\partial \bar{z}^l}}$ and such that $A1 = f$. This is the left multiplication operator by f with respect to $*$, $A = L_f$. In particular, one can immediately check that

$$L_{\nu \frac{\partial \Phi}{\partial z^k}} = \nu \left(\frac{\partial \Phi}{\partial z^k} + \frac{\partial}{\partial z^k} \right).$$

Now, for any formal function g on U we recover the product of f and g as $f * g = L_f g$. The local star products parametrized by ω agree on the intersections of coordinate charts and define a global star product on M.

We call the star product with separation of variables parametrized by the trivial deformation $\omega = \frac{1}{\nu}\omega_{-1}$ of ω_{-1} **standard**.

Explicit formulas for star products with separation of variables on pseudo-Kähler manifolds can be given in terms of graphs encoding the bidifferential operators C_r (see [11], [5], [12]).

In this paper we give a closed formula expressing the total symbol of the left star multiplication operator L_f of the standard star product with separation of variables $*$ on a coordinate chart U of a pseudo-Kähler manifold M in terms of a family of differential operators on the cotangent bundle T^*U acting on symbols of differential operators on U. One can immediately recover a formula for the star product $f * g$ on U from the total symbol of the operator L_f.

2. A recursive formula for the symbol of the left multiplication operator

A differential operator A on a real n-dimensional manifold M can be written in local coordinates $\{x^i\}$ on a chart $U \subset M$ in a normal form,

$$A = p_{i_1 i_2 \ldots i_n}(x) \left(\frac{\partial}{\partial x^1} \right)^{i_1} \ldots \left(\frac{\partial}{\partial x^n} \right)^{i_n},$$

where summation over repeated indices is assumed. Denote by $\{\xi_i\}$ the dual fibre coordinates on T^*U. Then the total symbol of A is given by the fibrewise polynomial function

$$\tau(A)(x,\xi) = p_{i_1 i_2 \ldots i_n}(x)\, (\xi_1)^{i_1} \ldots (\xi_n)^{i_n}$$

on T^*U. The mapping $A \mapsto \tau(A)$ is a bijection of the space of differential operators on U onto the space of fibrewise polynomial functions on the cotangent space T^*U. The composition of differential operators induces via this bijection an associative operation $\circ$ on the fibrewise

polynomial functions on T^*U. The composition $\circ$ of fibrewise polynomial functions $p(x,\xi)$ and $q(x,\xi)$ is given by the formula

$$(p\circ q)(x,\xi) = \exp\left(\frac{\partial}{\partial \eta_i}\frac{\partial}{\partial y^i}\right) p(x,\eta)q(y,\xi)\Big|_{y=x,\eta=\xi} =$$

$$\sum_{r=0}^{\infty}\frac{1}{r!}\frac{\partial^r p}{\partial \xi_{i_1}\dots\partial\xi_{i_r}}\frac{\partial^r q}{\partial x^{i_1}\dots\partial x^{i_r}}, \tag{1}$$

where the sum has a finite number of nonzero terms. If $p = p(x)$ or $q = q(\xi)$, then $p\circ q = pq$, which means that the operation $\circ$ has the separation of variables property with respect to the variables x and ξ. Formula (1) is valid for complex coordinates as well.

Now let $*$ be the standard star product with separation of variables on a pseudo-Kähler manifold (M,ω_{-1}) of complex dimension m. Choose a contractible coordinate chart $(U,\{z^k,\bar z^l\})$ on M and let Φ_{-1} be a potential of ω_{-1} on U. Given a formal function $f = f_0+\nu f_1+\dots$ on U, the left star multiplication operator L_f is the formal differential operator on U determined by the conditions that (i) $L_f 1 = f * 1 = f$, (ii) it commutes with the pointwise multiplication operators $R_{\bar z^l} = \bar z^l$, and (iii) it commutes with the operators

$$R_{\frac{\partial\Phi_{-1}}{\partial\bar z^l}} = \frac{\partial\Phi_{-1}}{\partial\bar z^l} + \nu\frac{\partial}{\partial\bar z^l}$$

for $1\leq l\leq m$. Also, the operator L_f is natural, i.e., $L_f = A_0+\nu A_1+\dots$, where A_r is a differential operator on U of order not greater than r.

Denote by $\{\zeta_k,\bar\zeta_l\}$ the dual fibre coordinates on T^*U. We want to describe conditions (i) - (iii) on the operator L_f in terms of its total symbol $F = \tau(L_f) = F_0+\nu F_1+\dots$, where $F_r = \tau(A_r)$. Condition (ii) means that F does not depend on the antiholomorphic fibre variables $\bar\zeta_l$, $F = F(\nu,z,\bar z,\zeta)$. Condition (i) means that $F|_{\zeta=0} = f$ and $F_r|_{\zeta=0} = f_r$. Condition (iii) is expressed as follows:

$$F\circ\left(\frac{\partial\Phi_{-1}}{\partial\bar z^l}+\nu\bar\zeta_l\right) = \left(\frac{\partial\Phi_{-1}}{\partial\bar z^l}+\nu\bar\zeta_l\right)\circ F. \tag{2}$$

Using the definition (1) of the operation $\circ$ and its separation of variables property we simplify (2):

$$F\circ\frac{\partial\Phi_{-1}}{\partial\bar z^l}+\nu\zeta_l F = \frac{\partial\Phi_{-1}}{\partial\bar z^l}F+\nu\zeta_l F+\nu\frac{\partial F}{\partial\bar z^l}. \tag{3}$$

We will use the conventional notation,

$$g_{k_1\dots k_r\bar l} = \frac{\partial^{r+1}\Phi_{-1}}{\partial z^{k_1}\dots\partial z^{k_r}\partial\bar z^l}.$$

Using (1) we simplify (3) further:

$$\sum_{r=1}^{\infty} \frac{1}{r!} g_{k_1 \ldots k_r \bar{l}} \frac{\partial^r F}{\partial \zeta_{k_1} \ldots \partial \zeta_{k_r}} = \nu \frac{\partial F}{\partial \bar{z}^l}. \tag{4}$$

In particular, $g_{k\bar{l}}$ is the metric tensor corresponding to ω_{-1}. We denote its inverse by $g^{\bar{l}k}$ and introduce the following operators:

$$\Gamma_r = g_{k_1 \ldots k_r \bar{l}}\, g^{\bar{l}k} \zeta_k \frac{\partial^r}{\partial \zeta_{k_1} \ldots \partial \zeta_{k_r}} \text{ and } D = \nu g^{\bar{l}k} \zeta_k \frac{\partial}{\partial \bar{z}^l}.$$

In particular,

$$\Gamma_1 = \zeta_k \frac{\partial}{\partial \zeta_k}$$

is the Euler operator for the holomorphic fibre variables. Multiplying both sides of (4) by $g^{\bar{l}k} \zeta_k$ and summing over the index l, we obtain the formula

$$\sum_{r=1}^{\infty} \frac{1}{r!} \Gamma_r F = DF. \tag{5}$$

We want to assign a grading to the variables ν and ζ_k such that $|\nu| = 1$ and $|\zeta_k| = -1$. Denote by $\mathcal{E}_p$ the space of formal series in the variables ν and ζ_k with coefficients in $C^\infty(U)$ such that the grading of each monomial $f(z, \bar{z}) \nu^r \zeta_{k_1} \ldots \zeta_{k_s}$ in such a series satisfies $r - s \geq p$. The spaces $\mathcal{E}_p$ form a descending filtration on the space $\mathcal{E} := \mathcal{E}_0$:

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \ldots.$$

Since L_f is a natural operator, its total symbol $F = \tau(L_f)$ is an element of $\mathcal{E}$. The operator Γ_r acts on $\mathcal{E}$ and raises the filtration by $r - 1$. The operator D acts on $\mathcal{E}$ and respects the filtration. Observe that the series on the left-hand side of (5) converges in the topology induced by the filtration on $\mathcal{E}$. The space $\mathcal{E}$ breaks into the direct sum of subspaces, $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, where $\mathcal{E}'$ consists of the elements of $\mathcal{E}$ that do not depend on the fibre variables ζ_k, i.e., $\mathcal{E}' = C^\infty(U)[[\nu]]$, and $\mathcal{E}''$ is the kernel of the mapping $\mathcal{E} \ni H \mapsto H|_{\zeta=0}$. Observe that the Euler operator $\Gamma_1 : \mathcal{E} \to \mathcal{E}$ respects the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, $\mathcal{E}'$ is its kernel, and $\mathcal{E}''$ is its image. Moreover, the operator Γ_1 is invertible on $\mathcal{E}''$. Every operator $\Gamma_k : \mathcal{E} \to \mathcal{E}$ maps $\mathcal{E}$ to $\mathcal{E}''$ and has $\mathcal{E}'$ in its kernel.

The following lemma is straightforward.

Lemma 1. *The operator* $\exp D = \sum_{r=0}^{\infty} \frac{1}{r!} D^r$ *acts on* $\mathcal{E}$ *and* $\exp(-D)$ *is its inverse operator on* $\mathcal{E}$*. The operator* $\exp D$ *leaves invariant the subspace* $\mathcal{E}''$ *and the operator* $\exp D - 1$ *maps* $\mathcal{E}$ *to* $\mathcal{E}''$*.*

Lemma 2. *We have the following identity,*

$$\Gamma_1 - D = \mathrm{e}^D\, \Gamma_1 \mathrm{e}^{-D}.$$

Proof. The lemma follows from the fact that $[\Gamma_1, D] = D$ and the calculation

$$\mathrm{e}^D\, \Gamma_1 \mathrm{e}^{-D} = \sum_{r=0}^{\infty} \frac{1}{r!} (\mathrm{ad}\, D)^r\, \Gamma_1 = \Gamma_1 - D.$$

□

Using Lemma 2, we rewrite formula (5) as follows:

$$\left(\mathrm{e}^D\, \Gamma_1 \mathrm{e}^{-D} + \sum_{r=2}^{\infty} \frac{1}{r!} \Gamma_r\right) F = 0. \tag{6}$$

Introduce the operator

$$Q = -\mathrm{e}^{-D} \left(\sum_{r=2}^{\infty} \frac{1}{r!} \Gamma_r\right) \mathrm{e}^D \tag{7}$$

on $\mathcal{E}$. It raises the filtration on $\mathcal{E}$ by one and maps $\mathcal{E}$ to $\mathcal{E}''$. Applying the operator $\exp(-D)$ on both sides of (6) we obtain that

$$(\Gamma_1 - Q)\, \mathrm{e}^{-D} F = 0. \tag{8}$$

Using the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ and the last statement of Lemma 1, we observe that $\exp(-D)F = f + H$ for some $H \in \mathcal{E}''$. We can rewrite formula (8) as follows:

$$(\Gamma_1 - Q)\, H = Qf. \tag{9}$$

Since the operator Q maps $\mathcal{E}$ to $\mathcal{E}''$ and Γ_1 is invertible on $\mathcal{E}''$, the operator $\Gamma_1^{-1} Q$ is well defined on $\mathcal{E}$ and raises the filtration by one, we obtain from (9) that

$$\left(1 - \Gamma_1^{-1} Q\right) H = \Gamma_1^{-1} Q f. \tag{10}$$

The operator $1 - \Gamma_1^{-1} Q$ is invertible and its inverse is given by the convergent series

$$\left(1 - \Gamma_1^{-1} Q\right)^{-1} = \sum_{r=0}^{\infty} \left(\Gamma_1^{-1} Q\right)^r.$$

We have

$$F = \mathrm{e}^D(f+H) = \mathrm{e}^D\left(f + \left(\sum_{r=0}^{\infty}\left(\Gamma_1^{-1}Q\right)^r\right)\Gamma_1^{-1}Qf\right) =$$

$$\mathrm{e}^D\left(\sum_{r=0}^{\infty}\left(\Gamma_1^{-1}Q\right)^r\right)f = \mathrm{e}^D\left(1-\Gamma_1^{-1}Q\right)^{-1}f.$$

Combining these arguments we arrive at the following theorem.

Theorem 1. *Given the standard star product with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}), a coordinate chart U on M, and a function $f \in C^\infty(U)[[\nu]]$, then the total symbol $F = \tau(L_f)$ of the left star multiplication operator by f is given by the following explicit formula,*

$$F = e^D\left(1-\Gamma_1^{-1}Q\right)^{-1}f. \tag{11}$$

Now, to find the star product $f * g$, one has to calculate the total symbol F of the operator L_f using formula (11), recover L_f from F, and apply it to g, $f * g = L_f g$.

One can use the same formula (11) to express the total symbol of the left multiplication operator L_f of the star product with separation of variables $*_\omega$ corresponding to an arbitrary formal deformation ω of the pseudo-Kähler form ω_{-1}. To this end one has to modify the operators Γ_r and D as follows. On a contractible coordinate chart U find a formal potential $\Phi = \frac{1}{\nu}\Phi_{-1} + \Phi_0 + \ldots$ of the form ω and set

$$G_{k_1\ldots k_r\bar{l}} := \frac{\partial^{r+1}\Phi}{\partial z^{k_1}\ldots\partial z^{k_r}\partial\bar{z}^l}.$$

Then $G_{k_1\ldots k_r\bar{l}} = \frac{1}{\nu}g_{k_1\ldots k_r\bar{l}} + \ldots$. Denote the inverse of $G_{k\bar{l}}$ by $G^{\bar{l}k} = \nu g^{\bar{l}k} + \ldots$. Now modify Γ_r and D (retaining the same notations) as follows:

$$\Gamma_r = G_{k_1\ldots k_r\bar{l}}G^{\bar{l}k}\zeta_k\frac{\partial^r}{\partial\zeta_{k_1}\ldots\partial\zeta_{k_r}} \text{ and } D = G^{\bar{l}k}\zeta_k\frac{\partial}{\partial\bar{z}^l}.$$

The Euler operator Γ_1 will not change. Define the operator Q by the same formula (7) with the modified Γ_r and D. Observe that we get the old operators Γ_r, D, and Q for the trivial deformation $\omega = \frac{1}{\nu}\omega_{-1}$. One can show along the same lines that formula (11) with the modified operators D and Q will be given by a convergent series in the topology induced by the filtration on $\mathcal{E}$ and will define the total symbol of the left star multiplication operator L_f with respect to the star product $*_\omega$.

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